Exam Analysis on Manifolds

[redacted]IANVAR-07.2019-2020.1B

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This exam consists of five assignments. You get 10 points for free.

Assignment 1. (5+10=15 pt.)
Let V be a real vector space of dimension four, and let $\omega \in A_2(V)$. Furthermore, let σ_1, σ_2 be linearly independent elements of $A_1(V)$.

1. Suppose there are $\tau_1, \tau_2 \in A_1(V)$ such that $\omega = \sigma_1 \wedge \tau_1 + \sigma_2 \wedge \tau_2$. Prove that $\omega \wedge \sigma_1 \wedge \sigma_2 = 0$.

2. Suppose
$$\omega \wedge \sigma_1 \wedge \sigma_2 = 0. \tag{1}$$
Prove that there are $\tau_1, \tau_2 \in A_1(V)$ such that
$$\omega = \sigma_1 \wedge \tau_1 + \sigma_2 \wedge \tau_2. \tag{2}$$
Hint: extend $\{\sigma_1, \sigma_2\}$ to a basis for $A_1(V)$, and use this basis to determine a basis for $A_2(V)$.

Assignment 2. (10+10=20 pt.)
The maps $\phi_1, \phi_2 : \mathbb{R} \to \mathbb{R}$ are given by $\phi_1(t) = t$ (so ϕ_1 is the identity map) and $\phi_2(t) = t^{1/3}$. Let $\mathcal{A}_i$, $i = 1, 2$, be the maximal atlas on $\mathbb{R}$ containing the chart $(\mathbb{R}, \phi_i)$.

1. Prove that the differentiable structures on $\mathbb{R}$ defined by $\mathcal{A}_1$ and $\mathcal{A}_2$ are different.

2. Let M_i, $i = 1, 2$, be the manifold $\mathbb{R}$ equipped with the atlas $\mathcal{A}_i$. Prove that M_1 and M_2 are diffeomorphic.

Assignment 3. (15 pt.)
Let $M = \{(x^1, x^2) \in \mathbb{R}^2 \mid \frac{1}{4} < (x^1)^2 + (x^2)^2 \leq 1\}$, endowed with the topology of the ambient space $\mathbb{R}^2$. Prove that M is a C^∞-manifold with boundary by constructing a C^∞-atlas for M.

Assignment 4. (12+8=20 pt.)
In this assignment we consider a *closed* one-form on $\mathbb{S}^n$. Here $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, as usual.

1. Prove that ω is exact if $n \geq 2$. (Hint: use an atlas on $\mathbb{S}^n$ consisting of two charts determined by stereographic projection from a pair of antipodal points on $\mathbb{S}^n$.)

2. Prove that ω is not necessarily exact if $n = 1$.

Assignment 5. (6+6+8=20 pt.)
Let $\mathbb{B}^n(r)$ be the n-dimensional closed ball of radius r, centered at the origin of $\mathbb{R}^n$, and let the $(n-1)$-sphere $\mathbb{S}^{n-1}(r)$ of radius r be its boundary. The radially outward pointing unit vector field of $\mathbb{S}^{n-1}(r)$ is

$$X = \frac{1}{r}\sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}.$$

Let $\Omega = dx^1 \wedge \cdots \wedge dx^n$ be the volume form on $\mathbb{R}^n$.

1. Let $V_n(r)$ be the volume of $\mathbb{B}^n(r)$, so $V_n(r) = \int_{\mathbb{B}^n(r)} \Omega$.
   Prove that $V_n(r) = r^n V_n(1)$.

   Hint: prove that the map $F : B_n(1) \to B_n(r)$, defined by $F(x) = rx$, is a diffeomorphism. Express $F^*\Omega$ in Ω, and prove that F is orientation preserving.

2. Prove that the volume form $\iota_X\Omega$ on $\mathbb{S}^{n-1}(r)$ is given by

$$\iota_X\Omega = \frac{1}{r}\sum_{i=1}^{n}(-1)^{\overset{i+1}{\cancel{i}}} x^i dx^1 \wedge \cdots \wedge \widehat{dx^i} \wedge \cdots \wedge dx^n.$$

   (Remark: formally, the volume form on $\mathbb{S}^{n-1}(r)$ is $i^*(\iota_X\Omega)$, where $i : \mathbb{S}^{n-1}(r) \to \mathbb{R}^n$ is the inclusion map. We don't distinguish between $i^*(\iota_X\Omega)$ and $\iota_X\Omega$.)

3. Let $A_{n-1}(r)$ be the volume of $\mathbb{S}^{n-1}(r)$, so $A_{n-1}(r) = \int_{\mathbb{S}^{n-1}(r)} \iota_X\Omega$. Prove that

$$A_{n-1}(r) = \frac{n}{r}V_n(r).$$

   Verify this identity for $n = 2$ and $n = 3$, given your knowledge of the circumference of a circle, etc.